

Explicit Solution of a Tropical Optimization Problem with Application to Project Scheduling

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Abstract

We examine an optimization problem in the tropical mathematics setting, which is motivated by practical problems in just-in-time scheduling. The problem is to minimize a nonlinear objective function defined through a multiplicative conjugate transposition on vectors in a finite-dimensional semimodule over a general idempotent semifield. To solve the problem, we first derive a lower bound for the objective function, and then find vectors that make the objective function equal to the bound. Under fairly general conditions, an explicit solution is given in a closed vector form. The result is applied to solve scheduling problems formulated as minimizing a function in the form of the span seminorm, subject to linear constraints. As an illustration, numerical examples of optimal scheduling are presented.

Key-Words: idempotent semifield, tropical optimization problem, nonlinear objective function, span seminorm, project scheduling.

MSC (2010): 65K10, 15A80, 65K05, 90C48, 90B35

1 Introduction

Tropical (idempotent) mathematics is concerned with the theory and applications of semirings with idempotent addition. Tropical mathematics has its origin in seminal works [1, 2, 3, 4, 5], where it is introduced as a natural tool to represent and solve real-world problems in operations research, including scheduling problems examined in [2, 3]. Over the past few decades, there has been significant progress in the field, which is reflected in several monographs and a wide range of research papers, among them most recent monographs [6, 7, 8, 9, 10, 11].

Since an early researches [12, 13], optimization problems that are defined and solved within the framework of tropical mathematics constitute an important research domain in the field. The problems are usually formulated

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as to minimize (maximize) functions on finite-dimensional semimodules over idempotent semifields, subject to constraints in the form of linear equalities and inequalities. Both linear and nonlinear objective functions are considered.

One of the objective functions, which is encountered in the problems, is the span (range) seminorm, which is given, in the ordinary setting, by the maximum deviation between components of a vector. This function is used as an optimality criterion for some problems in a range of areas from the analysis of Markov decision processes [14, 15] to the form-error measurement in precision metrology [16, 17]. In the context of tropical mathematics, the span seminorm is introduced by [18, 19], where it is called the range seminorm.

The span seminorm appears in [20, 21] in a tropical optimization problem coming from machine scheduling. The problem deals with machines, each producing a component for final products, and aims at finding starting time of production for each machine so that the completion time of all machines is spread over a shortest possible period of time. A solution to the problem is given in a form that involves both semifields $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\min,+}$.

In this paper, we consider a more general problem of just-in-time scheduling [22, 23]. It is formulated in the common setting of project scheduling as having a set of activities operating under various precedence relations between their initiation and completion times. The problem is to design a schedule that provides, as far as possible, a single common completion time of all activities, and thus can be solved by minimizing a span seminorm.

We represent the precedence relations in terms of the semifield $\mathbb{R}_{\max,+}$ by linear vector equalities and inequalities. The span seminorm is written in a straightforward, if not linear, vector form. As a result, we arrive at a constrained optimization problem with a nonlinear objective function, which involves a conjugate transposition operator, and with linear constraints.

The above problem is taken as a motivating example and a starting point to define and solve a new tropical optimization problem in a rather formal setting. We exploit the fact that the application of the solution for linear inequalities in [24, 10] allows to reduce the problem to an unconstrained problem with new variables. We examine an extended version of the unconstrained problem formulated in terms of a general idempotent semifield. To solve the latter problem, the solution approach developed in [25, 10, 26] is implemented based on the derivation of a sharp lower bound for the objective function.

We give an explicit solution to the extended problem under fairly general conditions, which is represented in a compact vector form in terms of the carrier semifield. The above scheduling problems are then solved as particular cases. Specifically, an alternative direct solution is suggested to the machine scheduling problem in [20, 21].

The paper is organized as follows. It begins with a motivating problem

drawn from just-in-time scheduling in Section 2. Furthermore, we give a brief introduction to basic definitions, notation, and preliminary results in tropical mathematics in Section 3 to provide a formal framework for subsequent results. Section 4 suggests the main results and includes definition and solution of general optimization problems with nonlinear objective functions. Application of the results to optimal scheduling problems are presented in Section 5, accompanied with illustrative numerical examples.

2 Motivating example

We start with a real-world problem taken from project scheduling and intended to both motivate and illustrate further results. The problem arises in just-in-time manufacturing and aims to design a schedule that minimizes the maximum deviation between completion time of activities in a project, subject to various activity precedence constraints. For more details and references on project scheduling, and specifically on just-in-time scheduling, see, e.g., [22, 23].

Consider a project which consists of n activities (jobs, tasks) operating under start-finish and start-start precedence constraints. The start-finish constraints require that a minimal time lag be held between initiation of one activity and completion of another. Each activity is assumed to complete as early as possible to meet the constraints. The start-start constraints specify a minimal time lag between initiation of any two activities. The problem is to find a schedule that provides, as far as possible under the constraints, a single common completion time for all activities.

For each activity $i = 1, \dots, n$, let x_i be an initiation time, y_i be a completion time, and c_{ij} be a minimum possible time lag between initiation of activity $j = 1, \dots, n$ and completion of i . Given time lags c_{ij} , the completion time of activity i must satisfy the start-finish precedence relations

$$x_j + c_{ij} \leq y_i, \quad j = 1, \dots, n,$$

with at least one inequality holding as equality. Note that we assume $c_{ii} \geq 0$ for all i . Provided that c_{ij} is not given for some j , we put $c_{ij} = -\infty$.

Now we combine the relations into one equality of the form

$$\max_{1 \leq j \leq n} (x_j + c_{ij}) = y_i.$$

Furthermore, let d_{ij} be given minimum possible time lag between initiation of activity j and initiation of i . Once again, we assume $d_{ij} = -\infty$ if no lag is specified for i and j . Due to the start-start constraints, we have relations

$$x_j + d_{ij} \leq x_i, \quad j = 1, \dots, n,$$

and then rewrite them as one inequality

$$\max_{1 \leq j \leq n} (x_j + d_{ij}) \leq x_i.$$

Finally, we define an objective function to formulate the design of optimal schedule as an optimization problem. An optimality criterion, which naturally shows if a schedule provides a single common completion time for all activities, is the maximum deviation between the completion times, or the span seminorm

$$\max_{1 \leq i \leq n} y_i - \min_{1 \leq i \leq n} y_i = \max_{1 \leq i \leq n} y_i + \max_{1 \leq i \leq n} (-y_i).$$

Now we can formulate an optimization problem of interest as follows. Given a_{ij} and b_{ij} for all $i, j = 1, \dots, n$, find x_1, \dots, x_n that

$$\begin{aligned} & \text{minimize} && (\max_{1 \leq i \leq n} y_i + \max_{1 \leq i \leq n} (-y_i)), \\ & \text{subject to} && \max_{1 \leq j \leq n} (x_j + c_{ij}) = y_i, \\ & && \max_{1 \leq j \leq n} (x_j + d_{ij}) \leq x_i, \quad i = 1, \dots, n. \end{aligned} \tag{1}$$

Note that a particular case of the problem is examined and solved in [20, 21] in the framework of tropical mathematics. A solution is given in a combined form that requires using both semifields $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\min,+}$.

Below we show how the problems can be represented in a different tropical mathematics setting and then directly solved in a compact vector form.

3 Preliminary results

In this section we give a brief overview of main algebraic definitions, notation and preliminary results, which provide a basis for subsequent solution of tropical optimization problems and their applications to project scheduling.

Both concise introductions and thorough presentations of the theory and methods of tropical mathematics are given in various forms in a range of works, including [6, 7, 9, 8, 27, 11]. Below we mainly adhere to results in [24, 10], which offer a useful framework for getting direct solutions in a compact form. For additional details, one can consult other publications listed above.

3.1 Idempotent semifield

Let \mathbb{X} be a set that is closed under two associative and commutative operations, addition \oplus and multiplication \otimes , and equipped with their neutral elements, zero 0 and identity 1 . Addition is idempotent, which implies that $x \oplus x = x$ for all $x \in \mathbb{X}$. Multiplication is distributive over addition

and invertible, that is, each $x \in \mathbb{X}_+$, where $\mathbb{X}_+ = \mathbb{X} \setminus \{0\}$, has inverse x^{-1} to satisfy $x^{-1} \otimes x = \mathbb{1}$. Since \mathbb{X}_+ forms a group under multiplication, the structure $\langle \mathbb{X}, 0, \mathbb{1}, \oplus, \otimes \rangle$ is commonly referred to as the idempotent semifield.

The integer power is introduced as usual. For any $x \in \mathbb{X}_+$ and integer $p > 0$, there are defined $x^0 = \mathbb{1}$, $0^p = 0$, $x^p = x^{p-1} \otimes x$, and $x^{-p} = (x^{-1})^p$.

In what follows, the multiplication sign \otimes is dropped for simplicity. The power notation is used in the sense of the above definition.

The idempotent addition produces a partial order, by which $x \leq y$ if and only if $x \oplus y = y$. The partial order is assumed to extend to a consistent total order over \mathbb{X} . From here on, the relation symbols and the minimization problems are thought in the context of this order.

As examples of the general semifield under consideration, one can take $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle$, $\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, + \rangle$, $\mathbb{R}_{\max,\times} = \langle \mathbb{R}_+ \cup \{0\}, 0, 1, \max, \times \rangle$, and $\mathbb{R}_{\min,\times} = \langle \mathbb{R}_+ \cup \{+\infty\}, +\infty, 1, \min, \times \rangle$, where \mathbb{R} is the set of reals, $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$.

Specifically, the semifield $\mathbb{R}_{\max,+}$ has the null $0 = -\infty$ and identity $\mathbb{1} = 0$. Each $x \in \mathbb{R}$ gets its inverse x^{-1} given by $-x$ in standard notation. For any $x, y \in \mathbb{R}$, the power x^y is equal to the arithmetic product xy . The order, which is induced by addition, corresponds to the natural linear order on \mathbb{R} .

3.2 Matrix algebra

Now we take into consideration matrices over \mathbb{X} and denote the set of matrices with m rows and n columns by $\mathbb{X}^{m \times n}$. A matrix with all entries equal to 0 is called a zero matrix. A matrix is row (column) regular, if it has no zero rows (columns). A matrix is regular, if it is both row and column regular.

For any conforming matrices $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$, $\mathbf{C} = (c_{ij})$, and scalar x matrix addition, matrix and scalar multiplication are routinely defined as

$$\{\mathbf{A} \oplus \mathbf{B}\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{\mathbf{B} \oplus \mathbf{C}\}_{ij} = \bigoplus_k b_{ik} c_{kj}, \quad \{x\mathbf{A}\}_{ij} = xa_{ij}.$$

For any matrix \mathbf{A} , its transpose is denoted by \mathbf{A}^T .

For any nonzero matrix $\mathbf{A} = (a_{ij}) \in \mathbb{X}^{m \times n}$, we introduce a multiplicative conjugate transpose $\mathbf{A}^- \in \mathbb{X}^{n \times m}$ with elements $a_{ij}^- = a_{ji}^{-1}$ if $a_{ji} \neq 0$, and $a_{ij}^- = 0$ otherwise, for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

Let us turn to square matrices in $\mathbb{X}^{n \times n}$. A matrix that has all diagonal entries equal to $\mathbb{1}$ and off-diagonal entries to 0 is an identity matrix denoted by \mathbf{I} . For any matrix \mathbf{A} , its trace is given by

$$\text{tr } \mathbf{A} = \bigoplus_{i=1}^n a_{ii}.$$

The matrices that have only one column (row) are routinely referred to as column (row) vectors. We denote the set of column vectors of order n by \mathbb{X}^n .

A vector having all components equal to $\mathbb{0}$ is a zero vector. A vector is regular if it has no zero components.

Let \mathbf{x} be a regular column vector and \mathbf{A} be a matrix. It is not difficult to see that the vector \mathbf{Ax} is regular only when the matrix \mathbf{A} is row regular. Similarly, the row vector $\mathbf{x}^T \mathbf{A}$ is regular only when \mathbf{A} is column regular.

As usual, a vector \mathbf{y} is linearly dependent on vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ if there are scalars $c_1, \dots, c_m \in \mathbb{X}$ such that $\mathbf{y} = c_1 \mathbf{x}_1 \oplus \dots \oplus c_m \mathbf{x}_m$. Specifically, a vector \mathbf{y} is collinear with \mathbf{x} if it holds that $\mathbf{y} = c\mathbf{x}$ for some scalar c .

For any nonzero vector $\mathbf{x} = (x_i) \in \mathbb{X}^n$, its multiplicative conjugate transpose is a row vector $\mathbf{x}^- = (x_i^-)$, where $x_i^- = x_i^{-1}$ if $x_i \neq \mathbb{0}$, and $x_i^- = \mathbb{0}$ otherwise. The next properties of conjugate transposition are easy to verify.

For any regular vectors \mathbf{x} and \mathbf{y} of the same size, the componentwise inequality $\mathbf{x} \leq \mathbf{y}$ implies $\mathbf{x}^- \geq \mathbf{y}^-$ and vice versa.

For any nonzero column vector \mathbf{x} , it holds that $\mathbf{x}^- \mathbf{x} = \mathbb{1}$. Moreover, if the vector \mathbf{x} is regular, then $\mathbf{x} \mathbf{x}^- \geq \mathbf{I}$.

3.3 Solution to linear inequality

Given a matrix $\mathbf{A} \in \mathbb{X}^{n \times m}$, consider a problem of finding regular solutions $\mathbf{x} \in \mathbb{X}_+^n$ that satisfy an inequality

$$\mathbf{Ax} \leq \mathbf{x}. \quad (2)$$

Below we present solutions given to the inequality in [24, 10] and written here in a more compact equivalent form.

For each matrix $\mathbf{A} \in \mathbb{X}^{n \times n}$, we introduce a function

$$\text{Tr}(\mathbf{A}) = \text{tr } \mathbf{A} \oplus \dots \oplus \text{tr } \mathbf{A}^n.$$

Furthermore, provided that $\text{Tr}(\mathbf{A}) \leq \mathbb{1}$, a star operator is defined to send \mathbf{A} to a matrix \mathbf{A}^* that consists of linearly independent columns of the matrix

$$\mathbf{I} \oplus \mathbf{A} \oplus \dots \oplus \mathbf{A}^{n-1}.$$

Lemma 1. *Let \mathbf{x} be the general regular solution of inequality (2). Then the following statements hold:*

1. *If $\text{Tr}(\mathbf{A}) \leq \mathbb{1}$, then $\mathbf{x} = \mathbf{A}^* \mathbf{u}$ for all regular vectors \mathbf{u} .*
2. *If $\text{Tr}(\mathbf{A}) > \mathbb{1}$, then there is no regular solution.*

4 Optimization problem

In this section we present our main result that solves an extended problem formulated in terms of a general idempotent semifield. We follow the solution approach based on deriving sharp bounds on the objective function in tropical optimization problems, which is introduced in [25] and then applied in a range of studies, including [10, 26].

Given matrices $\mathbf{A}, \mathbf{B} \in \mathbb{X}^{m \times n}$ and vectors $\mathbf{p}, \mathbf{q} \in \mathbb{X}^m$, the problem is to find regular vectors $\mathbf{x} \in \mathbb{X}^n$ that

$$\text{minimize } \mathbf{q}^- \mathbf{A} \mathbf{x} (\mathbf{B} \mathbf{x})^- \mathbf{p}. \quad (3)$$

The next statement offers a direct solution to the problem.

Theorem 2. *Suppose \mathbf{A} is column regular and \mathbf{B} is row regular matrices, \mathbf{p} is nonzero and \mathbf{q} is regular vectors. Denote $\Delta = (\mathbf{B}(\mathbf{q}^- \mathbf{A})^-)^- \mathbf{p}$.*

Then the minimum in problem (3) is equal to $\Delta > 0$ and attained at any vector

$$\mathbf{x} = \alpha(\mathbf{q}^- \mathbf{A})^-, \quad \alpha > 0.$$

Proof. To verify the statement, we first show that Δ is a lower bound for the objective function in (3), and then present vectors \mathbf{x} that provide the bound.

With inequality $\mathbf{x} \mathbf{x}^- \geq \mathbf{I}$, we write

$$\mathbf{q}^- \mathbf{A} \mathbf{x} \mathbf{x}^- \geq \mathbf{q}^- \mathbf{A}.$$

Since the vector \mathbf{q} is regular and the matrix \mathbf{A} is column regular, both sides in the last inequality are also regular and $\mathbf{q}^- \mathbf{A} \mathbf{x} > 0$ for any regular \mathbf{x} . Therefore, we have

$$(\mathbf{q}^- \mathbf{A} \mathbf{x})^{-1} \mathbf{x} = (\mathbf{q}^- \mathbf{A} \mathbf{x} \mathbf{x}^-)^- \leq (\mathbf{q}^- \mathbf{A})^-.$$

Multiplication by \mathbf{B} from the left gives

$$(\mathbf{q}^- \mathbf{A} \mathbf{x})^{-1} \mathbf{B} \mathbf{x} \leq \mathbf{B}(\mathbf{q}^- \mathbf{A})^-.$$

Considering that the matrix \mathbf{B} is row regular, both sides of the inequality are regular vectors, and thus

$$\mathbf{q}^- \mathbf{A} \mathbf{x} (\mathbf{B} \mathbf{x})^- \geq (\mathbf{B}(\mathbf{q}^- \mathbf{A})^-)^-.$$

After right multiplication of both sides by the vector \mathbf{p} , we finally have the lower bound in the form

$$\mathbf{q}^- \mathbf{A} \mathbf{x} (\mathbf{B} \mathbf{x})^- \mathbf{p} \geq (\mathbf{B}(\mathbf{q}^- \mathbf{A})^-)^- \mathbf{p} = \Delta > 0.$$

It remains to verify that $\mathbf{x} = \alpha(\mathbf{q}^- \mathbf{A})^-$ yields the bound for any $\alpha > 0$. Indeed, substitution into the objective function and identity $\mathbf{x}^- \mathbf{x} = \mathbf{1}$ give

$$\mathbf{q}^- \mathbf{A} \mathbf{x} (\mathbf{B} \mathbf{x})^- \mathbf{p} = \mathbf{q}^- \mathbf{A} (\mathbf{q}^- \mathbf{A})^- (\mathbf{B}(\mathbf{q}^- \mathbf{A})^-)^- \mathbf{p} = (\mathbf{B}(\mathbf{q}^- \mathbf{A})^-)^- \mathbf{p} = \Delta. \quad \square$$

We conclude this section with solutions of two particular cases of the above problem. First assume that $\mathbf{A} = \mathbf{B} = \mathbf{I}$ and $\mathbf{p} = \mathbf{q} = \mathbf{1}$, where $\mathbf{1}$ denotes a vector having all components equal to 1. We arrive at a problem that is to

$$\text{minimize } \mathbf{1}^T \mathbf{x} \mathbf{x}^{-1} \mathbf{1}.$$

Application of Theorem 2 immediately gives $\Delta = \mathbf{1}$ as the minimum in the problem, which is attained at any vector $\mathbf{x} = \alpha \mathbf{1}$ for all $\alpha > 0$.

Finally, we examine a problem that underlies the design of optimal schedules to be given below. We put $\mathbf{B} = \mathbf{A}$, $\mathbf{p} = \mathbf{q} = \mathbf{1}$, and consider a problem

$$\text{minimize } \mathbf{1}^T \mathbf{A} \mathbf{x} (\mathbf{A} \mathbf{x})^{-1} \mathbf{1}. \quad (4)$$

With Theorem 2, we immediately obtain the following result.

Corollary 3. *Suppose \mathbf{A} is a regular matrix and denote $\Delta = (\mathbf{A}(\mathbf{1}^T \mathbf{A})^{-1})^{-1} \mathbf{1}$.*

Then the minimum in problem (4) is equal to Δ and attained at any vector

$$\mathbf{x} = \alpha (\mathbf{1}^T \mathbf{A})^{-1}, \quad \alpha > 0.$$

5 Applications to optimal scheduling

We are now in a position to put the scheduling problem described above into the framework of tropical mathematics and then give a direct solution to the problem in a compact vector form.

5.1 Representation of scheduling problem

Consider problem (1) and note that its representation in ordinary notation involves only operations max, addition, and additive inversion. On this basis, it is natural to represent the problem in terms of the semifield $\mathbb{R}_{\max,+}$.

First we represent constraints in terms of $\mathbb{R}_{\max,+}$ as scalar equalities and inequalities

$$\bigoplus_{j=1}^n c_{ij} x_j = y_i, \quad \bigoplus_{j=1}^n d_{ij} x_j \leq x_i, \quad i = 1, \dots, n.$$

With matrix-vector notation

$$\mathbf{C} = (c_{ij}), \quad \mathbf{D} = (d_{ij}), \quad \mathbf{x} = (x_i), \quad \mathbf{y} = (y_i),$$

the scalar constraints take the form

$$\mathbf{C} \mathbf{x} = \mathbf{y}, \quad \mathbf{D} \mathbf{x} \leq \mathbf{x}.$$

Furthermore, we rewrite the objective function in (1). Since, for $\mathbb{R}_{\max,+}$, it holds that $\mathbb{1} = (0, \dots, 0)^T$, the objective function is readily given by

$$\left(\bigoplus_{i=1}^n y_i\right) \left(\bigoplus_{i=1}^n y_i^{-1}\right) = \mathbb{1}^T \mathbf{y} \mathbf{y}^{-1} \mathbb{1}.$$

Finally, by combining the objective function with the constraints, we arrive at a problem

$$\begin{aligned} & \text{minimize} && \mathbb{1}^T \mathbf{y} \mathbf{y}^{-1} \mathbb{1}, \\ & \text{subject to} && \mathbf{C} \mathbf{x} = \mathbf{y}, \\ & && \mathbf{D} \mathbf{x} \leq \mathbf{x}. \end{aligned} \tag{5}$$

5.2 Solution of scheduling problem

A direct solution to problem (5) under general conditions is given as follows.

Theorem 4. *Suppose that \mathbf{C} is a regular matrix and \mathbf{D} is a matrix with $\text{Tr}(\mathbf{D}) \leq \mathbb{1}$. Denote $\Delta = (\mathbf{C} \mathbf{D}^* (\mathbb{1}^T \mathbf{C} \mathbf{D}^*)^{-})^{-1} \mathbb{1}$.*

Then the minimum in problem (5) is equal to Δ and attained at

$$\mathbf{x} = \alpha \mathbf{D}^* (\mathbb{1}^T \mathbf{C} \mathbf{D}^*)^{-}, \quad \mathbf{y} = \alpha \mathbf{C} \mathbf{D}^* (\mathbb{1}^T \mathbf{C} \mathbf{D}^*)^{-}, \quad \alpha > 0.$$

Proof. It follows from Lemma 1 that the inequality constraints in problem (5) have the solution $\mathbf{x} = \mathbf{D}^* \mathbf{u}$ for all vectors $\mathbf{u} > 0$ of appropriate size.

With the solution, the equality constraints become $\mathbf{y} = \mathbf{C} \mathbf{D}^* \mathbf{u}$ with $\mathbf{u} > 0$.

Substitution of \mathbf{y} in the objective function of (5) leads to problem (3) with $\mathbf{A} = \mathbf{B} = \mathbf{C} \mathbf{D}^*$, $\mathbf{p} = \mathbf{q} = \mathbb{1}$, and an unknown regular vector \mathbf{u} .

Since $\mathbf{D}^* \geq \mathbf{I}$, the matrix $\mathbf{C} \mathbf{D}^*$ is regular. Application of Corollary 3 gives the minimum $\Delta = (\mathbf{C} \mathbf{D}^* (\mathbb{1}^T \mathbf{C} \mathbf{D}^*)^{-})^{-1} \mathbb{1}$ attained at $\mathbf{u} = \alpha (\mathbb{1}^T \mathbf{C} \mathbf{D}^*)^{-}$.

It remains to go back to the original vectors and get $\mathbf{x} = \alpha \mathbf{D}^* (\mathbb{1}^T \mathbf{C} \mathbf{D}^*)^{-}$ and $\mathbf{y} = \alpha \mathbf{C} \mathbf{D}^* (\mathbb{1}^T \mathbf{C} \mathbf{D}^*)^{-}$ for all $\alpha > 0$. \square

Finally consider a problem from [20, 21], that can now be represented as

$$\begin{aligned} & \text{minimize} && \mathbb{1}^T \mathbf{y} \mathbf{y}^{-1} \mathbb{1}, \\ & \text{subject to} && \mathbf{C} \mathbf{x} = \mathbf{y}. \end{aligned} \tag{6}$$

As a direct consequence of the above results, we get the following solution.

Corollary 5. *Suppose \mathbf{C} is a regular matrix and denote $\Delta = (\mathbf{C} (\mathbb{1}^T \mathbf{C})^{-})^{-1} \mathbb{1}$.*

Then the minimum in problem (6) is equal to Δ and attained at

$$\mathbf{x} = \alpha (\mathbb{1}^T \mathbf{C})^{-}, \quad \mathbf{y} = \alpha \mathbf{C} (\mathbb{1}^T \mathbf{C})^{-}, \quad \alpha > 0.$$

5.3 Numerical examples

To illustrate the results obtained, we consider an example project of three activities under constraints given by matrices

$$\mathbf{C} = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix},$$

where the symbol $0 = -\infty$ is used for ease of exposition.

First we do not take into account the start-start constraints so as to solve reduced problem (6). After calculating the vectors

$$\mathbb{1}^T \mathbf{C} = (4 \quad 3 \quad 3), \quad (\mathbb{1}^T \mathbf{C})^- = \begin{pmatrix} -4 \\ -3 \\ -3 \end{pmatrix}, \quad \mathbf{C}(\mathbb{1}^T \mathbf{C})^- = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

we apply Corollary 5 and immediately arrive at a solution

$$\Delta = 0, \quad \mathbf{x} = \alpha \begin{pmatrix} -4 \\ -3 \\ -3 \end{pmatrix}, \quad \mathbf{y} = \alpha \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha > 0.$$

Note that in this situation we really get a just-in-time schedule with a single common completion time of all activities.

Let us now incorporate the start-start constraints given by \mathbf{D} into the problem. We take \mathbf{D} to calculate

$$\mathbf{D}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & -3 & 0 \end{pmatrix}, \quad \mathbf{D}^3 = \begin{pmatrix} -1 & -2 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}, \quad \text{Tr}(\mathbf{D}) = 0,$$

and then evaluate the sum

$$\mathbf{I} \oplus \mathbf{D} \oplus \mathbf{D}^2 = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 2 \\ -1 & -3 & 0 \end{pmatrix}.$$

Since the first and the third columns in the obtained matrix are collinear, we drop the last column to define the matrix \mathbf{D}^* . We successively get

$$\mathbf{D}^* = \begin{pmatrix} 0 & -2 \\ 1 & 0 \\ -1 & -3 \end{pmatrix}, \quad \mathbf{C}\mathbf{D}^* = \begin{pmatrix} 4 & 2 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}, \quad \mathbb{1}^T \mathbf{C}\mathbf{D}^* = \begin{pmatrix} -4 \\ -3 \end{pmatrix}.$$

Furthermore, we have vectors

$$\mathbf{D}^*(\mathbb{1}^T \mathbf{C}\mathbf{D}^*)^- = \begin{pmatrix} -4 \\ -3 \\ -5 \end{pmatrix}, \quad \mathbf{C}\mathbf{D}^*(\mathbb{1}^T \mathbf{C}\mathbf{D}^*)^- = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.$$

Application of Theorem 4 gives the results

$$\Delta = 2, \quad \mathbf{x} = \alpha \begin{pmatrix} -4 \\ -3 \\ -5 \end{pmatrix}, \quad \mathbf{y} = \alpha \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \quad \alpha > 0.$$

The solution offers an schedule that is optimal with respect to the span seminorm. Note, however, that the constraints in the problem give no way for the schedule to provide a single common completion time of all activities.

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